Unified Engineering Fall 2007
Signals & Systems Problems

Assigned: Wednesday, November 21
Due: Friday, November 30
Note: Please hand in problems on separate sheets of paper

Problem S7.1: Look-back to Lectures 9, 10, 11 (15 points)

The satellite and scientific probe for the Gravity Probe-B experiment (launched in 2004) can be modeled as shown in the schematic below. The mass m_1 represents the mass of the spacecraft plus helium tank. The mass m_2 is the probe. The coupling between the probe and spacecraft is modeled using a spring with spring constant k and a dashpot with damping coefficient c. A force $f(t)$ is applied to the spacecraft body as shown. The displacement of the spacecraft and probe with respect to the inertial reference are respectively $d_1(t)$ and $d_2(t)$.

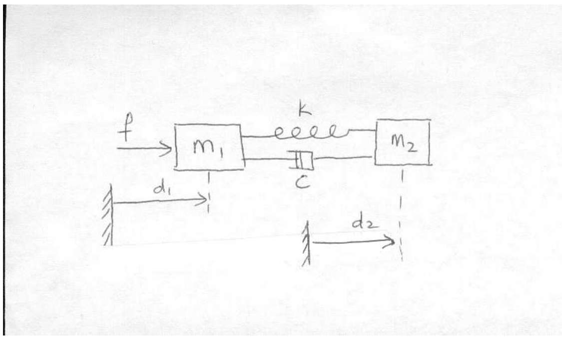

Figure 1: Schematic for the Gravity Probe-B model, Question S7.1.

(a) Derive the differential equations governing this system.

(b) If the input is the applied force $f(t)$ and the output of interest is the probe position $d_2(t)$, derive the state-space matrices A, B, C, and D, using state variables d_1, $\dot{d}_1$, d_2, and $\dot{d}_2$.

(c) The motion of the probe is expected to be very small, so your team lead prefers you to use the following state variables: $d_1' = 10^6 d_1$, $\dot{d}_1'$, $d_2' = 10^6 d_2$, $\dot{d}_2'$. What are the state-space matrices A', B', C', and D' for this choice of state?

(d) What is the transformation matrix T that relates the state in part (c) to the state in part (b), i.e.

$$\vec{x}\,' = T\vec{x}?$$

Show that the matrices you obtained in part (c) could also be obtained through the relationships derived in Lecture S11, i.e.

$$A' = TAT^{-1},\ \ B' = TB, C' = CT^{-1}, D' = D.$$

(e) For the choice of state

$$\vec{x}'' = \begin{bmatrix} d_1 & \dot{d}_1 & (d_2 - d_1) & (\dot{d}_2 - \dot{d}_1) \end{bmatrix}^T,$$

give the transformation matrix T and the state-space matrices A'', B'', C'', and D''.

(f) For the values $m_1 = 2000$kg, $m_2 = 1000$kg, $k = 3.2 \times 10^6$Nm, and $c = 4.6 \times 10^3$Nm/s, compute the eigenvalues of your state dynamics matrices A, A', and A''.

(g) For each eigenvalue (or each complex pair of eigenvalues) write one sentence explaining the physical meaning of the eigenvalue with reference to the satellite-probe system.

Problem S7.2: Look-ahead to Lectures S14 and S15 (15 points)

You saw in SL9 that the Quanser roll subsystem acts largely as a lightly damped second-order system. To simplify the analysis here, we will ignore the damping, and model the Quanser roll dynamics as a second-order undamped system. The state equations in state-space form therefore have the form:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\vec{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -w_n^2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ k \end{bmatrix}}_{B} u,$$

where the states have been chosen to be the roll angle and the roll rate, i.e. $x_1 = \phi$ and $x_2 = \dot{\phi}$. The input $u(t)$ is the voltage.

We will use a typical value for the Quanser natural frequency of $\omega_n = 1$ rad/s. The constant k takes a value $k = 1$.

(a) Compute the eigenvectors, v_1 and v_2, and the eigenvalues, λ_1 and λ_2, of the state dynamics matrix A.

(b) One possible choice for a state transformation is to use the eigenvectors of A. Use the following state transformation:

$$\vec{x} = V\vec{x}',$$

where $V = [v_1 \; v_2]$ is the matrix whose columns contain the eigenvectors of A. (So we have chosen $V = T^{-1}$, using the notation in Lecture S11.) Compute the transformed state matrices A' and B'. (Hint: if you first normalize your eigenvectors, then since the eigenvectors form an orthonormal set, V^{-1} can be easily computed as the complex conjugate transpose of V.)

(c) What do you notice about the form of your transformed matrix A'?

(d) We wish to compute the state response of the Quanser to a step input in voltage, i.e.

$$v(t) = \begin{cases} 0, & t < 0 \\ \bar{v}, & t > 0, \end{cases}$$

where $\bar{v}$ is some constant voltage value. The Quanser is initially at rest, i.e. $x_1(0) = 0$, $x_2(0) = 0$. First, compute the response of the transformed state $\vec{x}'(t)$. (Hint: you will first need to transform the initial conditions to the new state.) Then transform your resulting solution for $\vec{x}'(t)$ to obtain $\vec{x}(t)$.

(e) Verify that your solution for $\vec{x}(t)$ in part (d) satisfies the original state-space system and given initial conditions.

Extra Practice Problems

1. Sketch the block diagrams for your three state-space systems in Question S7.1.
2. Sketch the block diagrams for your two state-space systems in Question S7.2.

More questions will be posted over the weekend...